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NONLINEAR EQUATIONS OF MOTION OF AN EXTENSIBLE UNDERGROUND PIPELINE: DERIVATION AND NUMERICAL MODELING

V. A. Rukavishnikov and O. P. Tkachenko

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Slow motion of a pipeline modeled by a bent rod in a viscous medium is studied. It is assumed that the displacements of the axial line of the rod are finite and its strains are small. The mutual influence of the tensile axial force and transverse deflections is taken into account. Equations of motion are derived and some numerical examples are considered. An approximate estimate of stresses in the pipeline wall is given.

Key words: *curved pipeline, viscous medium, finite displacements, numerical analysis.*

Introduction. It is well known that the complex configuration of an underground pipeline can change owing to the action of an internal fluid flow, initial bending of the axis, and properties of the medium. This is a slow process resulting in considerable displacements of the axial line. Therefore, an underground pipeline should be designed with allowance for its motion under the action of three factors mentioned above.

Various formulations and solutions of the problem of motion of a pipeline with a fluid flow inside can be found in [1–6].

In the present paper, we study slow motion of a curved pipeline modeled by a bent rod under the action of viscous-fluid flow and nonlinear resistance of the external medium. Finite displacement are allowed, whereas strains are assumed to be small. Equations of motion are derived with allowance for the axial tensile force T produced by a transverse displacement of the pipeline. Oscillations of the fluid flow are ignored because the phenomenon studied has a different time scale.

1. Physical Formulation of the Problem. We consider a pipeline as an elastic hollow rod whose initial configuration is described by the equations of a plane curve $\Gamma_0 = \{x, y\colon\ x = x_0(s),\ y = y_0(s)\}$, where s is the arc length.

The rod is immersed in a strongly viscous medium and loaded by a steady fluid flow with a velocity v_0. In the initial state, no external forces act and internal stresses in the rod are equal to zero. Since the external medium is considered as a viscous fluid, the rod starts to move after loading. The rod ends are assumed to be clamped. It is required to determine its motion.

The following assumptions are used:

1) the strains of the rod are small, whereas the transverse displacements are finite but small compared to the rod length and the radius of curvature of the axial line Γ_0;

2) the tension of the rod produced by its bending is uniform along the rod;

3) for long periods of time, the external medium is described by the equations of a strongly viscous fluid;

4) the ratio $R_0/\min\rho_0$ is small (R_0 is the pipeline radius and $\min\rho_0$ is the minimum radius of curvature of the axis Γ_0);

5) the inertia of the wall and fluid is ignored.

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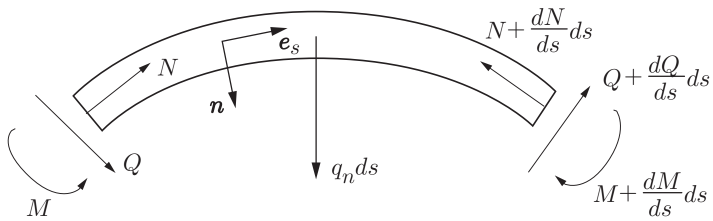

Fig. 1

2. Equations of Motion and Formulation of the Initial Boundary-Value Problem. We find the normal displacements of the rod w_n as a function of the time t and arc length s.

Let an element of a bent pipe of curvature $æ_0$ be loaded by the end and distributed forces (Fig. 1). In Fig. 1, Q is the transverse force, M is the bending moment, and N is the tensile force. We assume that the axial distributed loads (axial inertial forces and flow friction) except for the tensile force are negligible as compared to the transverse loads. In this case, the distributed load applied to the pipeline can be written in the form

$$q_n = -æ\rho_f S_f v_0^2 + q_{nr} + q_{ns},$$

where $æ(s,t)$ is the current curvature of the pipeline, v_0 is the velocity of the fluid flow, q_{nr} is the resistance force of the medium per unit length of the pipe, q_{ns} is the transverse force due to the tension of the pipeline per unit length, ρ_f is the density of the liquid, and S_f is the cross-sectional area of the flow.

Since the curvature and strains of the pipeline are small, we use the expression for the load produced by the axial tension:

$$q_{ns} = T\,\frac{\partial^2 w_n}{\partial s^2}.$$

We assume that, for small bending of the curve, the axial force T is constant along the pipeline and has the form [7]

$$T = \frac{ES_t}{l}\Big[\frac{1}{2}\int_0^l \Big(\frac{dy}{dx}\Big)^2 dx - \frac{1}{2}\int_0^l \Big(\frac{dy_0}{dx}\Big)^2 dx\Big]. \tag{1}$$

Here y and y_0 are the current and initial coordinates of the points of Γ, respectively, l is the distance between the fixed points of the pipeline, measured along the Ox axis, E is Young's modulus, and S_t is the cross-sectional area of the pipe. Given the transverse displacements of the pipe w_n, one can find the current coordinates of Γ using the formulas

$$x(s,t) = x_0(s) - \frac{dy_0}{ds}\,w_n(s,t), \qquad y(s,t) = y_0(s) + \frac{dx_0}{ds}\,w_n(s,t). \tag{2}$$

We note that the second integral in (1) does not vary and is equal to $l_0 - l$ (l_0 is the initial length of the pipeline measured along Γ_0).

Following [8], we obtain the equation of motion for the unknown displacement w_n. Differential equations of motion of the deformed rod have the form (Fig. 1)

$$\frac{\partial N}{\partial s} + æQ = 0, \qquad \frac{\partial Q}{\partial s} + æN - q_n = 0, \qquad Q = \frac{\partial M}{\partial s}.$$

Below, we use the well-known relations

$$M = EI(æ - æ_0), \qquad æ - æ_0 = \frac{\partial^2 w_n}{\partial s^2} + æ_0^2 w_n, \tag{3}$$

where I is the cross-sectional moment of inertia of the pipeline. Since the strains are small, we ignore the products of the desired function w_n and its derivatives. Expressing the current curvature $æ$ in terms of the initial curvature $æ_0$ and displacement of the wall w_n (3), after transformations we obtain

$$EI\Big(\frac{\partial^4 w_n}{\partial s^4} + 2æ_0^2\,\frac{\partial^2 w_n}{\partial s^2} + æ_0^4 w_n\Big) + \rho_f S_f v_0^2\Big(\frac{\partial^2 w_n}{\partial s^2} + æ_0^2 w_n + æ_0\Big) - q_{nr} - q_{ns} = 0. \tag{4}$$

To determine the resistance force of the medium q_{nr}, we consider the motion of an infinite cylinder in a viscous fluid and solve the Oseen equation [9]

$$(\boldsymbol{u}'\nabla)\boldsymbol{v} = -\frac{1}{\rho_{\text{soil}}}\,\nabla p + \nu\Delta\boldsymbol{v}, \qquad \nu = \frac{\mu}{\rho_{\text{soil}}}, \tag{5}$$

where p and $\boldsymbol{v}$ are the pressure and velocity of the medium, μ and ρ_{soil} are the viscosity and density of the medium, and $\boldsymbol{u}'$ is the velocity of the medium at infinity relative to the cylinder. Under the adhesion boundary conditions, the solution of Eq. (5) is known and the resistance force per unit length is given by

$$q_{nr} = -\frac{4\pi\mu u}{0.5 + \ln|4\mu/(\gamma\rho_{\text{soil}}R_0 u)|}, \tag{6}$$

where u is the transverse velocity of the cylinder and $\gamma = 1.7811$ is the Maskeroni number. Substituting (6) and q_{ns} into (4), we obtain the equations of motion of a bent pipeline in a viscous medium:

$$\begin{gathered} EI\,\frac{\partial^4 w_n}{\partial s^4} + (2EI\ae_0^2 + \rho_f S_f v_0^2 - T)\frac{\partial^2 w_n}{\partial s^2} + (EI\ae_0^2 + \rho_f S_f v_0^2)\ae_0^2 w_n \\ + \ae_0\rho_f S_f v_0^2 + \frac{4\pi\mu u}{0.5 + \ln|4\mu/(\gamma\rho_{\text{soil}}R_0 u)|} = 0, \\ u = \frac{\partial w_n}{\partial t}. \end{gathered} \tag{7}$$

Equations (7) are supplemented by the homogeneous initial and boundary conditions

$$\begin{gathered} w_n = 0 \qquad \text{for} \quad t = 0, \\ w_n = \frac{\partial w_n}{\partial s} = 0 \qquad \text{for} \quad s = 0, \quad s = l_0. \end{gathered} \tag{8}$$

System (7), (8) is a nonlinear initial boundary-value problem for the unknown functions w_n and u. The solution of system (7), (8) adequately describes the pipeline displacements, provided the displacements are small compared to the initial radius of curvature of the pipeline axis and the hypothesis of plane cross sections is valid. The first constraint is implied by formulas (3) and the second one, by the assumptions used to obtain the equations of [8].

3. Difference Scheme and Numerical Algorithm for Solving the Problem. To solve problem (1), (2), (7), (8), we use the two-layer explicit difference scheme for time

$$\begin{gathered} \frac{EI}{h_s^4}(w_{i+2}^j - 4w_{i+1}^j + 6w_i^j - 4w_{i-1}^j + w_{i-2}^j) \\ + (2EI\ae_{0i}^2 + \rho_f S_f v_0^2 - T^j)\,\frac{1}{h_s^2}\,(w_{i+1}^j - 2w_i^j + w_{i-1}^j) \\ + \left(EI\ae_{0i}^2 + \rho_f S_f v_0^2\right)\ae_{0i}^2 w_i^j + \ae_{0i}\rho_f S_f v_0^2 + \frac{4\pi\mu u_i^j}{0.5 + \ln(4\mu/(\gamma R_0 u_i^j \rho_{\text{soil}}))} = 0, \\ u_i^j = \frac{1}{h_t}\,(w_i^{j+1} - w_i^j), \qquad T^{j+1} = \frac{ES_t}{l}\Big[\frac{1}{2}\int\limits_0^l \Big[\Big(\frac{dy}{dx}\Big)_i^{j+1}\Big]^2 dx - (l_0 - l)\Big], \\ y_i^{j+1} = y_{0i} + \Big(\frac{dx_0}{ds}\Big)_i w_i^{j+1}, \qquad x_i^{j+1} = x_{0i} - \Big(\frac{dy_0}{ds}\Big)_i w_i^{j+1}. \end{gathered} \tag{9}$$

Here j enumerates time layers, i enumerates points along the coordinate s, h_s and h_t are the steps of the coordinate and time grids, respectively, u is the velocity of the pipe, and $\ae_{0i}$ is the initial curvature at the ith point. The boundary and initial conditions are taken into account in the usual fashion.

According to scheme (9), the calculations were performed as follows. For the known values of w_i^j at the jth layer, the velocities u_i^j were determined from the first difference equation using Newton's iterative method. Then the values of T^{j+1} and w_i^{j+1} were recalculated using the last formulas of (9). The time step that satisfied the stability and accuracy requirements was chosen by the trial-and-error method.

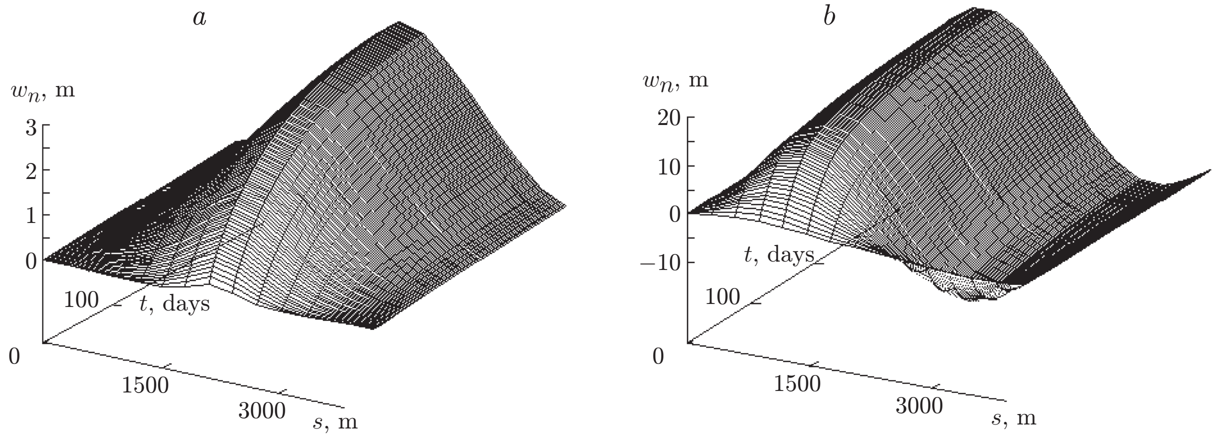

Fig. 2

4. Numerical Results. We consider two configurations of the pipeline:

$$y = C_1 \frac{A_1 - B_1(x-D)^2}{A_1 + B_1(x-D)^2} + C_1; \tag{10}$$

$$y = xB_2(x - A_2)(x - C_2). \tag{11}$$

Here $C_1 = 10$, $A_1 = 2000$, $B_1 = 0.2$, $D = 1500$, $B_2 = 0.000002$, $A_2 = 1500$, $C_2 = 3000$, and x varies from 0 to 3000 m. The characteristics of the external medium, fluid, and material of the wall are as follows: velocity of the fluid $v_0 = 1$ m/sec, density of the fluid $\rho_f = 800$ kg/m^3, density of the soil $\rho_{\text{soil}} = 1700$ kg/m^3, viscosity of the soil $\mu = 1000$ Pa$\cdot$sec^{-1}, thickness of the pipe wall $h = 0.005$ m, Young's modulus of the pipe material $E = 2 \cdot 10^{11}$ N/m^2, and pipe radius $R_0 = 0.3$ m.

Figure 2 shows the calculated displacement w_n versus time and arc length. In calculations, a period of a year was considered. In the case of profile (10) (Fig. 2a), the pipeline is initially deformed in the direction of action of the centrifugal forces and then in the direction of the maximum centrifugal force acting from the side of the fluid, until this force is balanced by the elastic forces.

In the case of profile (11) (Fig. 2b), the deflection increases in both directions owing to the symmetric profile. For profile (11), the maximum displacement is greater than that for profile (10). It is worth noting that the maximum initial curvatures of profiles (10) and (11) are $\text{æ}_{01} = 3.73 \cdot 10^{-3}$ m^{-1} and $\text{æ}_{02} = 1.22 \cdot 10^{-3}$ m^{-1}, respectively.

Variation in the fluid velocity v_0 and medium viscosity μ shows that the velocity u decreases with an increase in μ and increases with v_0.

Calculations were also performed for profiles (10) and (11) in which the tensile force was ignored. For profile (10), the results are very close to those obtained with allowance for the force T because the maximum displacement is small. In the case of profile (11), neglect of the tensile force leads to a 50-% increase in the maximum deflection. This result shows that the force T should be taken into account in the equations of motion. It is worth noting that no qualitative differences in the pipeline behavior are observed.

5. Estimate of Stresses in the Pipeline Wall. Above, we determined the normal displacement w_n as a function of coordinates and time. From the mechanical viewpoint, however, this does not mean that the problem is solved completely. For practical applications, it is necessary to estimate the maximum stresses that occur in the pipeline wall for the final configuration of the axial line. To this end, we assume that all stress-tensor components are negligible compared to the longitudinal stress σ_{ss}. The quantity $\max|\sigma_{ss}|$ is a sum of the bending stresses $\tilde{\sigma}_{ss}$ and tensile stresses $\bar{\sigma}_{ss}$. In this case, we obtain

$$\bar{\sigma}_{ss} = \frac{T}{S_t} = \frac{E}{l}\Big[\frac{1}{2}\int\limits_0^l \Big(\frac{dy}{dx}\Big)^2 dx - (l_0 - l)\Big].$$

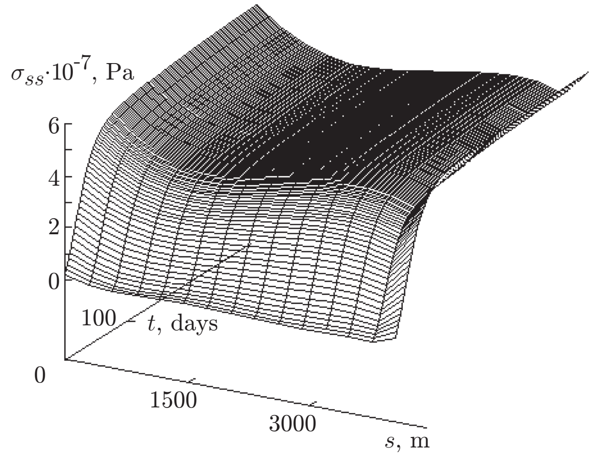

Fig. 3

To find $\tilde{\sigma}_{ss}$, we use the elementary theory of beam bending (see, e.g., [10])

$$\tilde{\sigma}_{ss} = E\tilde{y}/\rho \tag{12}$$

(ρ is the radius of curvature of the bent beam and $\tilde{y}$ is the distance from the neutral line to the point considered in the bending plane).

Let the pipeline be bent in the xOy plane. Using polar coordinates, we write $\tilde{y} = R\cos\theta$. In formula (12), we obtain

$$\frac{1}{\rho} = æ - æ_0 = \frac{\partial^2 w_n}{\partial s^2} + æ_0^2 w_n$$

(see, e.g., [8]).

Thus, the formula for stresses becomes

$$\sigma_{ss} = ER\cos\theta\Big(\frac{\partial^2 w_n}{\partial s^2} + æ_0^2 w_n\Big) + \frac{E}{l}\Big[\frac{1}{2}\int\limits_0^l \Big(\frac{dy}{dx}\Big)^2 dx - (l_0 - l)\Big]. \tag{13}$$

Figure 3 shows the stress versus the axial coordinate and time, calculated by formula (13) for $\cos\theta = 1$. The calculations show that the maximum stress for profile (11) is higher than that for profile (10). Moreover, $\Big|\max\limits_{0<s<l_0} \sigma_{ss}\Big| \leqslant 60$ MPa for profile (10). These stresses are much lower than the admissible stress $[\sigma] = 140$ MPa for the St. 3 steel. For carbon structural steels used in machine building, the admissible stress varies in the range 60–250 MPa (see, e.g., [11]). Consequently, even in the model problem, the stress approaches the critical value for certain materials. Since formula (13) is approximate, the problem of calculating stresses in the pipeline walls calls for a further detailed analysis.

6. Conclusions. A quasi-one-dimensional mathematical model of slow motion of a curved pipeline in a strongly viscous medium is proposed. It is shown that the equations of a strongly viscous fluid adequately describe the effect of soil. Test calculations are performed for two profiles of the pipeline and various physical parameters of the system. Approximate formulas for estimating the stresses in the pipeline wall are given.

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